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# STATISTICAL STUDIES IN STELLAR ROTATION II

## A Method of Analyzing Rotational Coupling in Double Stars and an Introduction to Its Applications

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STATISTICAL STUDIES IN STELLAR ROTATION II.

A Method of Analyzing Rotational Coupling in Double Stars and An Introduction to its Applications

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ABSTRACT

The coupling between the rotational velocities $\underline{v}_1$ and $\underline{v}_2$ of the components of double stars has bearing on their origin, on the problem of synchronism and on the distribution of angular momentum in dust and gas clouds. Since the observations give the apparent velocity $\underline{v}_k \sin \underline{i}_k$ $(k = 1, 2)$, the only large scale approach is statistical and it requires the knowledge of the probability density $\psi(\underline{i}_1, \underline{i}_2)$. When the existence of the equatorial break-up velocity $\underline{v}_{ok}$ is considered, one finds

$$\psi(i_1, i_2) = \sin i_1 \sin i_2 \int_0^{\sin i_1} dy_1 \int_0^{\sin i_2} dy_2\, F(y_1, y_2)\left[(1-y_1^2)(1-y_2^2)\right]^{-1/2}$$

where $F(y_1, y_2)$ is the observed frequency function of $y_1$ and $y_2$ defined by $y_k = \underline{v}_k \sin \underline{i}_k/\underline{v}_{ok}$. From the above equation the bivariate distribution $W(x_1, x_2)$ of $x_1$ and $x_2$, which are $\underline{v}_k/\underline{v}_{ok}$, is derived and the method of treating the observations is discussed.

The study of $W(x_1, x_2)$ for a sample of visual binaries indicates that the regression $\bar{x}_2(x_1)$ is nearly a linear function with $\bar{x}_2(0) = 0.2$ and $\bar{x}_2(1) = 0.8$ and that the array probability density $H(x_2 | x_1)$ is maximum for $x_1 = x_2$. A table of correspondence between $\underline{v}_1$ and the average value of $\underline{v}_2$ is given in the range BO-FO. The proposed method of analysis lends support to the existence of synchronism in closely spaced binaries.

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## I. Introduction

The study of the rotational correlation between components of double stars has a bearing on a variety of astrophysical problems. Among these the origin of binaries, their subsequent modes of evolution and the physical interactions in closely-spaced systems are of primary interest. The study may also give an insight into the distribution of angular momentum among pre-stellar blobs in gas and dust clouds. Whatever their origin, the present rotational status of short period detached and semi-detached systems is thought to reflect mainly physical processes such as tidal interactions (Abt [1970], Plavec [1970] and references therein), mass transfer (*cf.* Plavec [1970]) or evolution modes in the hydrogen burning stage (Van den Heuvel [1970]). Most of these stars are considered to be synchronized with the orbital motion (Plaut [1959], Olson [1968], Plavec [1970], Van den Heuvel [1970]). The inference of the existence of synchronism is based on the assumption that the spin axes are perpendicular to the orbital plane. The problem has four degrees of freedom. Whereas the above assumption appears to be interesting, it is certainly much too restrictive, *a priori*. The existence of a preferential angle between the spins may, however, not be independent of the early stages of evolution of a close binary as suggested by Plavec (1970).

Another interesting group of double systems is the one constituted by wide pairs for which the existence of some degree of coupling has been shown (Steinitz and Pyper [1970]). Wide pairs are perhaps the most suitable to be investigated in relation to their very early history since little or no interaction has occurred and the components have evolved as single stars. A quantitative evaluation of the difference in coupling between such pairs and short-period binaries may also tell us to what extent wide pairs have a different origin. Rotational coupling refers to correlation between the spin angular momenta. Since the observations give us the projected rotational velocity $\underline{v}$ sin $\underline{i}$ we should first study the correlation between the equatorial velocities $\underline{v}_1$ and $\underline{v}_2$ of the components by eliminating the aspect dependence.

The purpose of this paper is to consider this problem from a statistical standpoint. We first develop an interpretational theory of rotational correlation (Section II). The proposed method is then discussed with regard to its applicability to existing observations (Section III). The theory is applied in Sections IV and V to a sample of visual binaries, which have been the least studied for rotational coupling. Consideration of eclipsing systems and spectroscopic binaries is limited to show how the degrees of freedom in the spin parallelism problem can be reduced (Section VI). Section VII presents a summary and some final remarks.

## II. The Integral Equations Governing the Frequency Distributions of the Apparent and True Rotational Velocities

The observations give the projected rotational velocity $u_k = v_k \sin i_k$ ($k = 1,2$; omitted hereinafter). Therefore, a study of the correlation between the true velocities $v_1$ and $v_2$ requires the knowledge of the inclination to the line of sight $i_k$. Attempts to evaluate $i$ have been reviewed by Maeder (1971). Recently, Hardorp and Scholz (1971) have shown that for stars rotating near the critical velocity it may be possible to derive $i$ from the profiles of He I$\lambda$4471 and Mg II$\lambda$4481. Maeder (1971) has proposed a double-entry table which gives the value of $i$ provided that $v \sin i$ and the difference in magnitude $\Delta M_V$ due to rotation are known. The latter condition meets severe restrictions and the method fails for a random sample of field stars, as is also pointed out by him.

It cannot be excluded that future studies will lead, in particular cases, to an observed value of $i$. Presently, the only large-scale approach is statistical. It implies assuming a probability density $\psi(i_1, i_2)$ which permits to derive the probability density $W(v_1, v_2)$ from the observed frequency distribution $F(u_1, u_2)$ by means of integral equations.

It is customarily assumed that the axes of rotation of the stars are oriented at random in space, and the function

$$\psi_0(i) = \sin i \tag{1}$$

has been used for describing the frequency of occurrence of $i$. The use of relation (1) makes it straightforward to compute the mean value of $v$ and the higher moments of the distribution function $H(v)$ (Chandrasekhar and Münch [1950]). However, it has been shown (Bernacca [1970], hereinafter referred to as Paper I), that a more general frequency distribution for the inclination $i$ is given by

$$\psi(i) = \sin i \int_0^i \varphi(\theta)(\cos\theta)^{-1}d\theta \tag{2}$$

where $\varphi(\theta)$ gives the distribution of $\theta$, and $\theta$ is defined by

$$u = v \sin i = v_0 \sin\theta \tag{3}$$

where $v_0$ is the critical velocity (equatorial break-up) for the star for which $u$ has been observed. According to relation

(2) we can use relation (1) only if all stars do not rotate at all, or if their velocities can have *any* value, from zero to infinity.

Relation (1) also places a restriction on the analysis of the rotational properties of physical binaries. If we assume that

$$\psi_0(i_1, i_2)\, di_1\, di_2 = \sin i_1 \sin i_2\, di_1\, di_2 \qquad (4)$$

gives the fraction of binaries with $i_1$ and $i_2$ in the interval ($di_1$, $di_2$), we encounter two difficulties:

(a) the inclinations $i_k$ are assumed to be independent "*a priori*", which leads us to admit '*a priori*' that the angle between the spin axes is randomly distributed, and

(b) there is no possibility of distinguishing physical pairs from any artificial pair of stars, since relation (4) does not indicate whether or not to each primary we have attached its own secondary.

Suppose we have a sample of binaries whose apparent velocities $u_k$ have been observed. Let $v_{ok}$ be the cirtical velocity of the k-th component.

Since $v_k \leq v_{ok}$, relation (3) implies that

$$\theta_K \leq i_K \leq \pi/2$$

It follows that the number of binaries with primaries in the range $\underline{di_1}$ and secondaries in the range $\underline{di_2}$ is, for any pair of values $(\theta_1, \theta_2)$,

$$\psi(i_1, i_2 | \theta_1, \theta_2)\, di_1\, di_2 = \sin i_1 \sin i_2 [\cos\theta_1 \cos\theta_2]^{-1} di_1\, di_2 \quad (5)$$

The number of binaries in the interval $(d\theta_1, d\theta_2)$ is known, since $\theta_k$ is observed. Let this number be $\varphi(\theta_1, \theta_2)\, d\theta_1\, d\theta_2$. Then, the frequency distribution of $\underline{i}_1$ and $\underline{i}_2$ is given by

$$\psi_0(i_1, i_2) = \sin i_1 \sin i_2 \int_0^{i_1} d\theta_1 \int_0^{i_2} d\theta_2\, \varphi(\theta_1, \theta_2)(\cos\theta_1 \cos\theta_2)^{-1} \quad (6)$$

It is easily seen that relation (2) represents the marginal distribution of $\underline{i}_k$ by integrating (6) with respect to $\underline{i}_h$. If $\theta_1$ and $\theta_2$ are not correlated, that is, if the function $\varphi(\theta_1, \theta_2)$ can be written as the product of two univariate distributions $\varphi_1(\theta_1)$ and $\varphi_2(\theta_2)$, the function $\psi_o(\underline{i}_1, \underline{i}_2)$ degenerates to the product of two functions given by relation (2) as was expected.

By combining relations (2) and (6) we can derive the dependence of $\underline{i}_2$ upon $\underline{i}_1$ by means of the array distribution $\psi(\underline{i}_2 | \underline{i}_1)$

$$\psi(i_2 | i_1) = \frac{\psi_o(i_1, i_2)}{\psi_1(i_1)} \tag{7}$$

For the purpose of illustration we have plotted in Figure 1 the function $\psi(\underline{i}_2 | \underline{i}_1)$ for the case in which $\varphi(\theta_1, \theta_2)$ is

$$\varphi(\theta_1, \theta_2) = \frac{4}{\pi} \cos^2\theta_1 \, \delta(\theta_2 - \theta_1)$$

which is a result not inconceivable in stellar rotation. The dashed line shows the function (1) for comparison. A frequency function peaked toward large value of $\underline{v}$ sin $\underline{i}$, like that of Be stars (Paper I) would give an extremely tight correlation between $\underline{i}_2$ and $\underline{i}_1$ for companions having the same apparent velocity as the primary.

By considering now the relations

$$x_K = \frac{v_K}{v_{0K}} = \frac{\sin\theta_K}{\sin i_K}$$

we can obtain the frequency distribution $W(x_1, x_2)$ from relation (5) and the observed $\varphi(\theta_1, \theta_2)$ in the usual way (cf. Trumpler and Weaver 1953). We find

$$(x_1 x_2)^2 W(x_1, x_2) =$$

$$\int_0^{\sin^{-1}x_1} d\theta_1 \int_0^{\sin^{-1}x_2} d\theta_2 \; \frac{\sin^2\theta_1 \sin^2\theta_2 \, \varphi(\theta_1, \theta_2)(\cos\theta_1 \cos\theta_2)^{-1}}{(x_1^2 - \sin^2\theta_1)^{1/2}(x_2^2 - \sin^2\theta_2)^{1/2}}$$

The above equation can be written in terms of the frequency distribution $F(y_1, y_2)$ of $y_1$ and $y_2$ defined by $y_k = \underline{u}_k/\underline{v}_{ok}$, as

$$(x_1 x_2)^2 W(x_1, x_2) = \int_0^{x_1} dy_1 \int_0^{x_2} dy_2 \, (y_1 y_2)^2 F(y_1, y_2) R^{-1/2} \tag{8}$$

where $R = (x_1^2 - y_1^2)(x_2^2 - y_2^2)(1 - y_1^2)(1 - y_2^2)$.

Relation (6) and (2) can also be easily written in terms of $F(y_1, y_2)$ and the marginal distribution $F_k(y_k)$ respectively.

It appears that the knowledge of $W(x_1, x_2)$ through $\psi_o(\underline{i}_1, \underline{i}_2)$, or of $H_k(x_k)$ through $\psi_k(\underline{i}_k)$ depends on the location of the observer in space. This restriction is, however, of some utility. Suppose we wish to compare two or more samples of stars, each sample located in a different part of the galaxy. If there are sufficient reasons to believe that the distribution $H(\underline{v})$ is the same in each sample, different observed distributions $F(\underline{v} \sin \underline{i})$ may imply that $\psi(\underline{i})$ is different for each sample. Such an inference may give valuable information about the orientation in the galaxy of the spin axis of the stars. It can also be shown that the proposed equations take into account selection effects with respect to the inclination $\underline{i}$. Finally, the dependence of $x_k$ upon $x_h$ can be described by means of the array distribution $H(x_k \mid x_h)$ which is

$$H(x_k \mid x_h) = W(x_1, x_2)/H_h(x_k) \qquad (9)$$

The marginal distribution $H_h(x_h)$ is easily found to be (see also Paper I):

$$H_h(x_h) = \frac{1}{x_h^2} \int_0^{x_h} \frac{y_h^2 F_h(y_h)\, dy_h}{(1-y_h)^{1/2}(x_h^2 - y_h^2)^{1/2}} \tag{10}$$

In the next section we shall discuss how to apply relations (8) and (9) to actual observations.

## III. Discussion

Whatever the form of the function $\psi(i)$, the correct procedure for deriving $H(v)$ has been described by Chandrasekhar and Münch (1950). This procedure implies assuming a function $H(v;\, a_1 \ldots, a_n)$ with a number of parameters $a_1, \ldots, a_n$, and using the appropriate integral equation, fitting the computed distribution $F(u)$ to the observed histogram. If the observed distribution $F(u)$ has not a simple form, it may be difficult to decide *a priori*, which model function is most suitable for treating the problem. A family of univariate distributions $H(v)$ of the Van Dien (1948) type, is given by

$$H(v) = \text{const.} \left[ e^{-j^2 (v - v_m)^2} + e^{-j^2 (v + k v_m)^2} \right] \tag{11}$$

Here $\underline{k}$ is a third parameter in addition to $\underline{j}$ and $\underline{v}_m$. If $\underline{k} = 1$, relation (11) reduces to the function of Chandrasekhar and Münch (1950).

Deutsch (1970) has used bimodal Maxwellian distributions in the discussion of "Y" and "O" populations of rotating stars. His approach requires special attention because it is independent of any form for $\psi(\underline{i})$. The starting point of Deutsch's work is that if a vector $\vec{\underline{V}}$ is isotropic, the distribution of the scalar $\underline{V}$ is Maxwellian. The above result is based on the assumption that, when the vector $\vec{\Omega} = \underline{j\vec{V}}$ is decomposed into components along Catesian axes, the distribution of any component, say $\Omega_z$, is independent of the other two components ($\Omega_x$ and $\Omega_y$ in this case). The assumption makes it possible to write

$$H(\Omega_x, \Omega_y, \Omega_z) = h(\Omega_x^2)\, h(\Omega_y^2)\, h(\Omega_z^2) \tag{12}$$

with obvious meaning of the symbols. This implies that for any pair of values of, say, $\Omega_x{}^2$ and $\Omega_y{}^2$, we can have any value for $\Omega_z{}^2$. If we consider that $\Omega = \underline{jV} \leq \underline{jV_o}$, $\underline{V_o}$ being the critical velocity, it follows that a stochastic dependence exists among $\Omega_x$, $\Omega_y$ and $\Omega_z$ and that relation (12) no longer holds. A procedure for describing the rotational properties of the binaries is therefore suggested as follows.

First, we consider the histogram of the velocities $y_1 = x_1 \sin i_1$ of the primaries and derive the true distribution $H_1(x_1)$ using the fitting procedure previously described and the relation (Paper I):

$$F_1(y_1) = \frac{2}{\pi} \frac{(1-y_1^2)^{1/2}}{y_1} \frac{d}{dy_1} y_1 \int_0^{y_1} \frac{x_1 H_1(x_1)\, dx_1}{(y_1^2 - x_1^2)^{1/2}} \tag{13}$$

which is the solution of (10) with respect to $F_1(y_1)$. Now, relation (8) can be written as:

$$H(x_2 | x_1) = \frac{1}{x_1^2 H_1(x_1)} \int_0^{x_1} \frac{y_1^2 F_1(y_1)\, \Phi(x_2 | y_1)\, dy_1}{(1-y_1^2)^{1/2} (x_1^2 - y_1^2)^{1/2}} \tag{14}$$

which permits us to derive the conditioned distribution $H(x_2 | x_1)$ when $\Phi(x_2 | y_1)$ is known. The conditioned distribution

$\Phi(x_2 | y_1)$ may be found using relation (13) when $F_1(y_1)$ is replaced by the array distribution $F(y_2 | y_1)$, which is observed. The resulting $\Phi(x_2 | y_1)$ may however be affected by the uncertainty in the fitting owing to a generally small number of stars in each $y_1$-array, leading to an appreciable error in the evaluation of the integral (14).

By considering, then, the above $\Phi(x_2 | y_1)$ to be a first approximation result, we can derive the conditioned distribution $f(y_1 | x_2)$ by means of

$$H_2(x_2) f(y_1 | x_2) = F_1(y_1) \Phi(x_2 | y_1) \tag{15}$$

where $H_2(x_2)$ is the marginal distribution of the secondaries with respect to $x_2$, obtained in the same way as for $H_1(x_1)$. We finally have

$$f(y_1 | x_2) = \frac{2}{\pi} \frac{(1-y_1^2)^{1/2}}{y_1} \frac{d}{dy_1} y_1 \int_0^{y_1} \frac{x_1 H(x_1 | x_2) dx_1}{(y_1^2 - x_1^2)^{1/2}} \tag{16}$$

Now $H(x_1 | x_2)$ may be assumed to be a function of the type given in equation (11) and we can fit $f(y_1 | x_2)$ as computed through (16) to the same function given by (15). Finally, relation (9) solves the problem completely.

The method of analysis thus far developed depends on knowledge of the critical velocity $\underline{v}_o$ for each star. Values of $\underline{v}_o$ for a wide range of masses and spectral types are known (*cf*. Slettebak [1966], Hardorp and Scholz [1971]) and depend on the model chosen to represent a rotating star. The maximum observed *v* sin *i* in the Slettebak system are systematically lower at each spectral type than expected if the fastest rotators are edge-on stars near to the equatorial break-up. The work by Hardorp and Strittmatter (1968) and Hardorp and Scholz (1971) has shown that the discrepancy should be attributed to narrowing effects in the line profile when gravity darkening is neglected in the calibration of *v* sin *i*. The model values of $\underline{v}_o$ reduced to the Slettebak system agree well with the largest *v* sin *i* throughout the range B2-F0 (see Hardorp and Scholz [1971]). Thus, assuming the "corrected" value of $\underline{v}_o$ or the maximum observed velocity, $(v \sin i)_o$, as the upper limit to the true velocity, may be a matter of personal choice. In the following we assume $(v \sin i)_o$ to be the upper limit. Figure 2 presents an updated trend of the largest velocities in the Slettebak system as a function of spectral type for main-sequence single stars (visual systems included) and for spectroscopic binaries. Be stars, stars with shell characteristics, Ap, Am and Apec stars are excluded. The two curves have been derived from the Asiago Catalogue of Rotational Velocities (Bernacca and Perinotto [1970], [1971]) by the following procedure. Stars of spectral type B0, B1 and B2 have been grouped together and the three largest *v* sin *i* values have been averaged. The result is plotted at the spectral type B1. The next step considers stars of spectral types B1, B2 and B3, and the average value is plotted against B2. This procedure takes into account both the uncertainty of one tenth of spectral class and that *v* sin *i* for a star is affected by a large error. The curves shown in Figure 2 may be assumed to give a characteristic value of $(v \sin i)_o$ along the main sequence. In the range A3-A6 and F0-F7 the spectroscopic binaries have lower values of $(v \sin i)_o$ than single stars, while earlier than A2 there seems to be no difference. The result should not be considered as definitive since the Asiago Catalogue is biased in the binary stars section, insofar as it includes, in addition to well-established cases, also stars reported as suspected binaries. The values of the model critical velocities $v_o$ reduced to the Slettebak system by Hardorp and Scholz(1971) agree well with the solid curve in Figure 2 in the range B5-F0. The discrepancy earlier than B5 should be attributed to the lack of Be stars in the present curve. For practical purposes in treating visual binaries we shall assume the values of $(v \sin i)_o$ listed in Table 1, which have been approximated to the nearest high 25 km/sec.

## IV. Observational Evidence of Rotational Correlation in Visual Binaries

The correlation between $\underline{u}_1$ and $\underline{u}_2$ for 50 visual pairs has been recently examined by Steinitz and Pyper (1970). They found a linear correlation coefficient of 0.46 in contrast to a value of 0.001 for pairs formed by matching to each primary a secondary at random. According to the above authors the coupling between $\underline{u}_1$ and $\underline{u}_2$ does not depend on selection effects in the limiting magnitude and on the small difference in spectral type between the components. In Table 2, 34 visual double stars are presented. They have been taken from Slettebak (1963) according to the following criteria:

(a) Both components have spectral type earlier than F3 and the same luminosity class in the range IV-V.

(b) Be, shell stars, Ap, Am and Apec stars are excluded.

Figure 3 is a plot of $\underline{u}_2$ vs. $\underline{u}_1$. The straight lines are the regressions $\bar{\underline{u}}_1(\underline{u}_2)$ and $\bar{\underline{u}}_2(\underline{u}_1)$ given by the following relations

$$\bar{\underline{u}}_1 = 42 + (0.76 \pm 0.12)\ \underline{u}_2 \qquad \text{km/sec.}$$

$$\bar{\underline{u}}_2 = 35 + (0.74 \pm 0.12)\ \underline{u}_1 \qquad \text{km/sec.}$$

The correlation ratio is 0.75. The probability of having by chance a correlation ratio greater than or equal to 0.75 with

34 observations is of the order of $10^{-6}$. The correlation ratio for the fractional velocities $y_k = \underline{u}_k/\underline{v}_{ok}$ is 0.73. The tighter correlation found for the program sample with respect to the sample analyzed by Steinitz and Pyper is due to the omission of binaries with components of type Am and Ap.

The parameters of the distributions $F_1(y_1)$ and $F_2(y_2)$ are presented in Table 3. They indicate that the distribution of the secondaries is nearly symmetrical around the average of $y_2$ and flatter than a normal law. The distribution of the primaries appears to be more similar to a Gaussian as far as the peakedness is concerned but it is more skewed toward large values of $y_1$ than the distribution of the secondaries. On the other hand, both the mean values of $y_k$ and the variances are the same. It cannot be excluded that with a larger number of data we could be able to distinguish better between $F_1(y_1)$ and $F_2(y_2)$. In the following we shall assume a unique frequency distribution $F(y)$ for both components and we will use different indices for the purpose of notation only. This assumption will yield, of course, the same dependence of $x_2$ upon $x_1$ as for that of $x_1$ upon $x_2$, but it permits us to consider a sample of 68 stars.

## V. Coupling Between the Equatorial Velocities $v_1$ and $v_2$ in Visual Binaries

The observed frequency function of the single components (marginal distribution) is represented by the histogram in Figure 4. The blocks are rather large for the following reasons: First, an error of 10% in $\underline{u}$ and $\underline{v}_o$ yields an error of 20% in y. Secondly, for small values of y we have too few stars.

A simple solution is indicated by the dashed line in Figure 4. This is the marginal distribution $H_1(x_1) = H_2(x_2)$ with

$$H_1(x_1) = C e^{-\frac{1}{2}\left(\frac{x_1 - x_0}{s}\right)^2} \tag{17}$$

where $x_o = 0.5$, $s = 0.4$ and C is the normalization factor. The solid curve is the marginal distribution $F_1(y_1)$ computed using relation (17) and (13) by numerical methods. It is seen that $F_1(y_1)$ represents the main features of the histogram. The average value of $x_1$ is 0.5 and the dispersion is 0.26. The corresponding moments of $F_1(y_1)$ are 0.43 and 0.23

which may be compared with the observed values 0.45 and 0.26 (Table 3). The slight discrepancy is due to having neglected a number of rapid rotators in the fitting of $F_1(y_1)$ to the histogram. In order to have a function $F_1(y_1)$ within all the statistical errors (vertical bars in Figure 4) we would have to consider the general form (11) with a suitable value for k. This would imply the existence of two normal populations of rotating binaries and the size of the statistical errors would introduce a personal bias in estimating the fraction of one population to the other one. The distribution $H_1(x_1)$ given by (17) appears to be therefore acceptable within the family of unimodal frequency functions. According to the discussion in Section III, the conditioned distribution $\Phi(x_2 | y_1)$ is required as the next step. Figure 5 presents the histograms of the velocities $y_2$ for selected intervals of $y_1$. They represent the observed array distribution $F(y_2 | y_1)$. It is clear that with 68 components it is hard to make a conditioned histogram free from selection effects and with nicely small statistical errors. However, there seems to be some regularity as was expected. The dashed curves in Figure 5 are Gaussian laws as relation (17) and give the required $\Phi(x_2 | y_1)$. The solid lines are the function $F(y_2 | y_1)$, computed by means of relation (13), which may be considered to represent the observations with a reasonable degree of reliability.

Before to derive $f(y_1 | x_2)$ from relation (15) it seems correct to find a function $\phi(x_2 | y_1)$ most free from the particular choice of the intervals in $y_1$ used to build the histograms of Figure 5. In Figure 6 the parameters of $\phi(x_2 | y_1)$ have been therefore plotted against $y_1$ in order to establish the relations $x_o(y_1)$ and $s(y_1)$ and finally to use the function

$$\Phi(x_2 | y_1) = C(y_1)\, e^{-\frac{1}{2}\left(\frac{x_2 - x_o(y_1)}{s(y_1)}\right)^2} \tag{18}$$

in (15). If the problem is well conditioned by a large number of observations, $f(y_1 | x_2)$ would turn out to be automatically normalized for every $x_2$. This may not occur in a coarse fitting and we are forced to proceed by trial and error. In principle the fitting should involve also $F_1(y_1)$. In this application we fix $F_1(y_1)$ to be the one previously derived and we try to find suitable functions $x_o(y_1)$ and $s(y_1)$. The straight lines shown in Figure 6 ($s = 0.23$ and $x_o = 0.05 + 1.055\ y_1$) are the final relations which by means of (18), (17) and (15) give the required array distribution $f(y_1 | x_2)$. Three particular curves are shown in Figure 7 (solid lines) for $x_2 = 0.1$, $0.5$ and $0.9$. They are affected by a slight approximation but appear suitable to be treated by equation (16). As a final step we may assume the function $H(x_1 | x_2)$ to be a Guassian like (17) and fit $f(y_1 | x_2)$ obtained through (16) to that already found.

In Figure 7 the fitting is shown for $H(x_1 | 0.9)$ which is the worst case encountered. The open circle gives the computed $f(y_1 | 0.9)$ which is seen to match the solid curve satisfactorily well. The parameters of $H(x_1 | x_2)$ result to be $x_o = x_2$ and $s = 0.25$. The correlation between $x_1$ and $x_2$ is described by using the mode $x_{1M}$ of $H(x_1 | x_2)$, the mean of $x_1$ and the standard deviation $\sigma_{x_1}$. These quantities are given in Figure 8 as a function of $x_2$.

It is seen that the distribution $H(x_1 | x_2)$ is hetero-skedastic. Whereas the result appears interesting no conclusions can be drawn about the real existence of this particular kind of stochastic dependence since it depends on having assumed $s(y_1)$ = constant from Fig. 6.

The marginal distribution $H_k(x_k)$ and the bivariate $W(x_1, x_2)$ are given in Table 4.

## VI . Remarks on the Spin Parallelism Problem

Spectroscopic binaries and eclipsing systems could be, in principle, treated by the method of analysis thus far developed and illustrated, provided that a suitable value of $\underline{v}_o$ is chosen. We should emphasize that $\underline{v}_o$ is not necessarily the critical velocity. It can be any other upper limit truly recognized and Figure 2 shows that for spectrophotometric systems it may in fact be smaller than the critical value.

Since, however, the hypothesis of the spin perpendicular to the orbital plane is generally used, we should examine how the equations presented support the above inference. Let $H(x_1 | x_2)$ be the conditioned distribution of $x_1$ and $f(y_1 | x_2)$ that of $y_1$ for each $x_2$-array. Synchronism implies that $x_2 = x_1$ if the components have equal masses and radii. Therefore, we may use $H(x_1 | x_2) = \delta(x_1 - x_2)$ in (16). Then

$$f(y_1 | x_2) = 0 \qquad \text{for } y_1 < x_2 \tag{19}$$

which makes it necessary to have $y_1 = x_2$ and therefore $i_1 = \pi/2$. In the same way, using the function $\delta(x_2 - x_1)$ we can infer that $i_2 = \pi/2$. The conclusion is that we can predict the same velocity $x_k$ for both components if they have the same apparent velocity $y_k$. As a consequence, their spin axis are perpendicular to the line of sight. Since components of eclipsing systems are observed in most cases to have $y_1 = y_2$, it follows that $i_k = \pi/2$ and $x_1 = x_2$ is a possible solution. It is seen therefore that the spin parallelism problem is reduced to assuming only that the axes of rotation have the same aximuthal angle. Moreover since the orbital plane contains the line of sight (essentially) the assumption that both axes are perpendicular to that plane is better supported from an epistemologic point of view.

The situation sketched above may appear to be paradoxical, for one may not see a good reason to not have, say, $y_2 < x_2$, if $x_1 = x_2$. The fact is that the present method has a different nature from that based on the probability density $\sin i$. The reader is referred to Paper I.

## VII. Summary

a) The Theory

The method of studying statistically the rotation of the stars is based on the consideration that the equatorial velocities are limited by the critical velocity $v_o$ obtained when the centrifigal force balances the gravity at the equator.

The consequence is that the probability density function $\psi(i)$ which describes the frequency of occurrence of the inclination $i$ of the axis of rotation to the line of sight is

$$\psi(i) = \sin i \int_0^{\sin i} F(y)(1-y^2)^{-1/2} dy \qquad \text{(a)}$$

where $y = x \sin i$, and $x = v/v_o$, if $v$ is the linear velocity in km/sec. Here F(y) is the observed frequency distribution of y.

Relation (a) depends on the critical velocity $\underline{v}_o$ which is known for main sequence stars (Hardorp and Scholz 1971). Any upper limit to the true velocities smaller than $\underline{v}_o$ may however be assumed. In the case of the Ap stars a value of 100 km/sec is well representative of the largest speed they actually may have (Bernacca and Perinotto 1971).

For an analysis of the rotation of double stars we have

$$\psi(i_1, i_2) = \sin i_1 \sin i_2 \int_0^{\sin i_1} dy_1 \int_0^{\sin i_2} dy_2 \frac{F(y_1, y_2)}{(1-y_1^2)^{1/2}(1-y_2^2)^{1/2}} \qquad \text{(b)}$$

where $F(y_1, y_2)$ is the frequency distribution of the apparent velocities $y_k (k = 1, 2)$ of the components.

The bivariate frequency distribution $W(x_1, x_2)$ of the true velocities $x_k = \underline{v}_k/\underline{v}_{ok}$ can be readily derived by the procedure described in Section III.

b) Rotational Correlation in Visual Binaries

The apparent velocities $\underline{v}_1 \sin \underline{i}_1$ and $\underline{v}_2 \sin \underline{i}_2$ for 34 visual systems from the Slettebak (1963) paper give a correlation ratio of 0.75. The probability that this ratio is due to the chance is $10^{-6}$. The method of analysis applied to such a sample gives a bivariate frequency distribution $W(x_1, x_2)$

whose characteristics are the following. The centroid is at $x_1 = x_2 = 0.5$ and the standard deviations are $\sigma_1 = \sigma_2 = 0.26$. The mode $x_{2M}$ of the distribution of the secondaries for any given value of $x_1$ for the primaries is a linear relation $x_{2M} = x_1$; therefore, the probability is maximum for having equal velocities $x_1$ and $x_2$, or, if the stars have the same spectral type, equal velocities $\underline{v}_1$ and $\underline{v}_2$. The mean velocity $\bar{x}_2$ of the secondaries is nearly a linear function of the velocity $x_1$ of the primaries with $\bar{x}_2(0) = 0.2$ and $\bar{x}_2(1) = 0.8$. It may be that the sample studied is made up of two populations, slow and fast rotators. If this could be established on the basis of a richer sample, then the analysis of rotational coupling should be carried out within each group. In Table 5 we give the expected average value of $\underline{v}_2$ for a set of values of $\underline{v}_1$. The table is based on the regression line $\bar{x}_2(x_1)$ shown in Figure 8 and on Table 1. Table 5 may be used for visual systems having components of the same spectral type. An analogous table for any combination of spectral types can be easily derived by using Figure 8 and Table 1. We recall that the items in Table 5 are in the Slettebak system.

c) The Spin Parallelism Problem

The equations presented in this investigation reduce considerably the number of degree of freedom in the assumption that the spin axes of components of eclipsing systems are perpendicular to the orbital plane, thus lending support to the existing inferences about synchronism.

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I wish to thank Dr. S. Chandrasekhar for his criticism of the original manuscript of April 1969 upon which both Paper I and this investigation are based. I am indebted to Dr. D. Fischel who helped me in performing part of the numerical analysis and to Dr. S. Sobieski for helpful discussions.

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TABLE 1

Characteristic Maximum Rotational Velocities (km/sec) in the Slettebak System for Main-Sequence Single Stars

| Spectral Type | $(v \sin i)_o$ | Spectral Type | $(v \sin i)_o$ |
|---|---|---|---|
| B0-B1 | 300 | A6-A8 | 250 |
| B2-B3 | 325 | A9-F0 | 200 |
| B4-B5 | 400 | F1-F3 | 225 |
| B6 | 425 | F4 | 150 |
| B7-B9 | 375 | F5-F6 | 125 |
| A0-A2 | 350 | F7 | 100 |
| A3-A4 | 325 | F8 | 50 |
| A5 | 275 | F9 | 25 |

TABLE 2

Rotational Velocities (km/sec) of 34 Visual Systems Observed by Slettebak (1963)

| ADS (BDS) | | $m_v$ | Sp. Type | $\underline{v} \sin \underline{i}$ | ADS (BDS) | | $m_v$ | Sp. Type | $\underline{v} \sin \underline{i}$ |
|---|---|---|---|---|---|---|---|---|---|
| 824 | A | 6.0 | B9.5 V | 300 | 899 | A | 5.6 | A0 V | 250 |
| | B | 6.8 | A1 V | 250 | | B | 5.8 | B9 V | 250 |
| 1683 | A | 6.1 | B9 V | 200 | 2270 | A | 5.4 | B7 V | 200 |
| | B | 6.7 | A1 V | 300 | | B | 6.8 | B9 V | 200 |
| (1731) | A | 7.1 | A1 V | 100 | 2582 | A | 6.5 | A2 V | 100 |
| | B | 7.5 | A3 V | 250 | | B | 6.9 | A3 V | 100 |
| (2313) | A | 4.3 | B3 V | 150 | 3597 | A | 6.6 | B8 V | 350 |
| | B | 7.3 | A1 V | 100 | | B | 7.1 | B9 V | 350 |
| 3962 | A | 5.0 | B1 V | 350 | 4068 | A | 5.9 | B9 V | 150 |
| | B | 7.1 | B3 V: | 350 | | B | 6.7 | A0 V | 200 |
| 4182 | A | 4.7 | B0.5 V | ≤25 | 4262 | A | 6.8 | B7 V | 200 |
| | B | 5.7 | B1 V | ≤25 | | B | 7.4 | B9.5 V | 120 |
| 4773 | A | 6.1 | A2 V | 120 | 6255 | A | 4.5 | B6 V | 60 |
| | B | 6.9 | A5 V | 120 | | B | 4.7 | B7 V | 200 |
| 7979 | A | 4.5 | A0 V | 180 | 8630 | A | 3.7 | F0 V | ≤25 |
| | B | 6.4 | A1 V | 250 | | B | 3.7 | F0 V | ≤25 |
| (6498) | A | 6.7 | F2 V | ≤25 | 9247 | A | 5.1 | A0 V | 100 |
| | B | 7.1 | F3 V | ≤30 | | C | 6.8 | A9 V | 70 |
| 9258 | A | 6.4 | A2 V | 80 | 9277 | A | 7.0 | A0 V | 100 |
| | B | 7.0 | A4 V | 60 | | B | 7.5 | A0 V | 80 |
| 9 α Lib | | 2.9 | A3 V | 80 | 9474 | A | 6.8 | F0 IV | 80 |
| 8 α Lib | | 5.3 | F2 V | ≤25 | | B | 7.6 | F2 IV | ≤30 |
| 9701 | A | 4.2 | F0 IV | 80 | 10129 | A | 5.6 | B9 V | 220 |
| | B | 5.3 | F0 IV-V | 70 | | B | 6.6 | A0 V | 250 |
| 10149 | A | 5.7 | A0 V | 140 | 10750 | A | 6.3 | A0 IV-V | 90 |
| | B | 6.8 | A5 V | 80 | | B | 6.7 | A0 V | 120 |
| 11089 | A | 5.9 | A3 V | 180 | 11593 | A | 6.1 | B3 V | 130 |
| | B | 6.0 | A3 V | 180 | | B | 7.1 | B8 V | 150 |
| 11635 | A | 5.1 | A5 V | 200 | 11853 | A | 4.5 | A5 V | 130 |
| | B | 6.2 | A7 V | 150 | | B | 4.9 | A7 V | 220 |
| 11870 | A | 6.6 | A0 V | 100 | 13087 | A | 5.8 | B6 V | 360 |
| | B | 7.4 | A2 V | 150 | | B | 6.5 | B8 V | 250 |
| 13902 | A | 6.1 | A2 V | 350 | 15147 | A | 6.3 | A1 V | 100 |
| | B | 6.9 | A8 V | 150 | | B | 7.7 | F2 V | 30 |

TABLE 3

Frequency Distribution of the Fractional Velocity $y_k = v_k \sin i_k / v_{ok}$ of Visual Systems, Where $v_{ok}$ is the Critical Velocity

| | Primaries | Secondaries |
|---|---|---|
| Mean | 0.45 | 0.46 |
| St. deviation | 0.26 | 0.27 |
| Skewness | 0.58 | 0.07 |
| Kurtosis | -0.32 | -1.05 |

TABLE 4

Bivariate Frequency Distribution $W(x_1, x_2)$ of the True Velocities $x_k = \underline{v}_{ok} \sin \underline{i}_k / \underline{v}_{ok}$ of Visual Binaries

| $x_1$ \ $x_2$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.848 | 1.706 | 1.342 | 0.899 | 0.514 | 0.250 | 0.104 | 0.036 | 0.011 | 0.003 | 0.001 |
| 0.1 | 1.724 | 1.867 | 1.724 | 1.356 | 0.909 | 0.519 | 0.253 | 0.105 | 0.037 | 0.012 | 0.003 |
| 0.2 | 1.406 | 1.787 | 1.935 | 1.787 | 1.406 | 0.942 | 0.538 | 0.262 | 0.109 | 0.038 | 0.011 |
| 0.3 | 0.982 | 1.465 | 1.863 | 2.018 | 1.863 | 1.465 | 0.982 | 0.561 | 0.273 | 0.114 | 0.040 |
| 0.4 | 0.581 | 1.016 | 1.516 | 1.927 | 2.088 | 1.927 | 1.517 | 1.016 | 0.581 | 0.282 | 0.118 |
| 0.5 | 0.286 | 0.588 | 1.030 | 1.536 | 1.952 | 2.115 | 1.952 | 1.536 | 1.030 | 0.588 | 0.286 |
| 0.6 | 0.118 | 0.282 | 0.581 | 1.016 | 1.517 | 1.927 | 2.088 | 1.927 | 1.516 | 1.016 | 0.581 |
| 0.7 | 0.040 | 0.114 | 0.273 | 0.561 | 0.982 | 1.465 | 1.863 | 2.018 | 1.863 | 1.465 | 0.982 |
| 0.8 | 0.011 | 0.038 | 0.109 | 0.262 | 0.538 | 0.942 | 1.406 | 1.787 | 1.935 | 1.787 | 1.406 |
| 0.9 | 0.003 | 0.012 | 0.037 | 0.105 | 0.253 | 0.519 | 0.909 | 1.356 | 1.724 | 1.867 | 1.724 |
| 1.0 | 0.001 | 0.003 | 0.011 | 0.036 | 0.104 | 0.250 | 0.514 | 0.899 | 1.342 | 1.706 | 1.848 |
| $H_k(x_k)$ | 0.579 | 0.767 | 0.955 | 1.116 | 1.226 | 1.265 | 1.226 | 1.116 | 0.955 | 0.767 | 0.579 |

TABLE 5

Expected Average Value of the Rotational Velocities $\underline{v}_2$ (km/sec) of Secondaries of Visual Binaries as a Function of the Velocity $\underline{v}_1$ of the Primary. Both Components have the same spectral type.

| $\underline{v}_1$ (km/sec) | B0-B1 | B2-B3 | B4-B5 | B7-B9 | A0-A2 | A3-A4 | A6-A8 | A9-F0 |
|---|---|---|---|---|---|---|---|---|
| 50 | 80 | 85 | 100 | 95 | 90 | 85 | 70 | 65 |
| 100 | 115 | 125 | 130 | 125 | 120 | 125 | 105 | 100 |
| 150 | 150 | 155 | 165 | 160 | 155 | 155 | 145 | 115 |
| 200 | 185 | 190 | 200 | 195 | 190 | 190 | 180 | 160 |
| 250 | 220 | 225 | 235 | 230 | 230 | 225 | 200 | |
| 300 | 240 | 250 | 270 | 265 | 260 | 250 | | |
| 350 | | | 300 | 290 | 280 | | | |

## Captions to the Figures

Fig. 1. The conditioned distribution $\psi(\underline{i}_2 | \underline{i}_1)$ for $\underline{i}_1 = 30^\circ$, $60^\circ$, $90^\circ$ is shown by the solid curves in the case of a model distribution $\varphi(\theta_1, \theta_2) = 4/\pi \cos^2\theta_1 \cdot \delta(\theta_2 - \theta_1)$. The angle $\theta_k$ is defined by $\underline{v}_k \sin \underline{i}_k = \underline{v}_{ok} \sin \theta_k$, where $\underline{v}_k$ is the equatorial velocity and $\underline{v}_{ok}$ the break-up limit, (k = 1, 2). The dashed line is the probability density function sin i based on geometrical considerations only.

Fig. 2. Characteristic largest observed rotational velocities for main-sequence single stars (dots) and spectroscopic binaries (open circle). The latter is biased by inclusion of stars whose duplicity is not fully established.

Fig. 3. Plot of the apparent velocities of the secondaries ($\underline{u}_2 = \underline{v}_2 \sin \underline{i}_2$) *vs.* the apparent velocities of the primaries ($\underline{u}_1 = \underline{v}_1 \sin \underline{i}_1$) for 34 visual systems from the paper by Slettebak (1963). The straight lines are the regression lines. The correlation coefficient is 0.75.

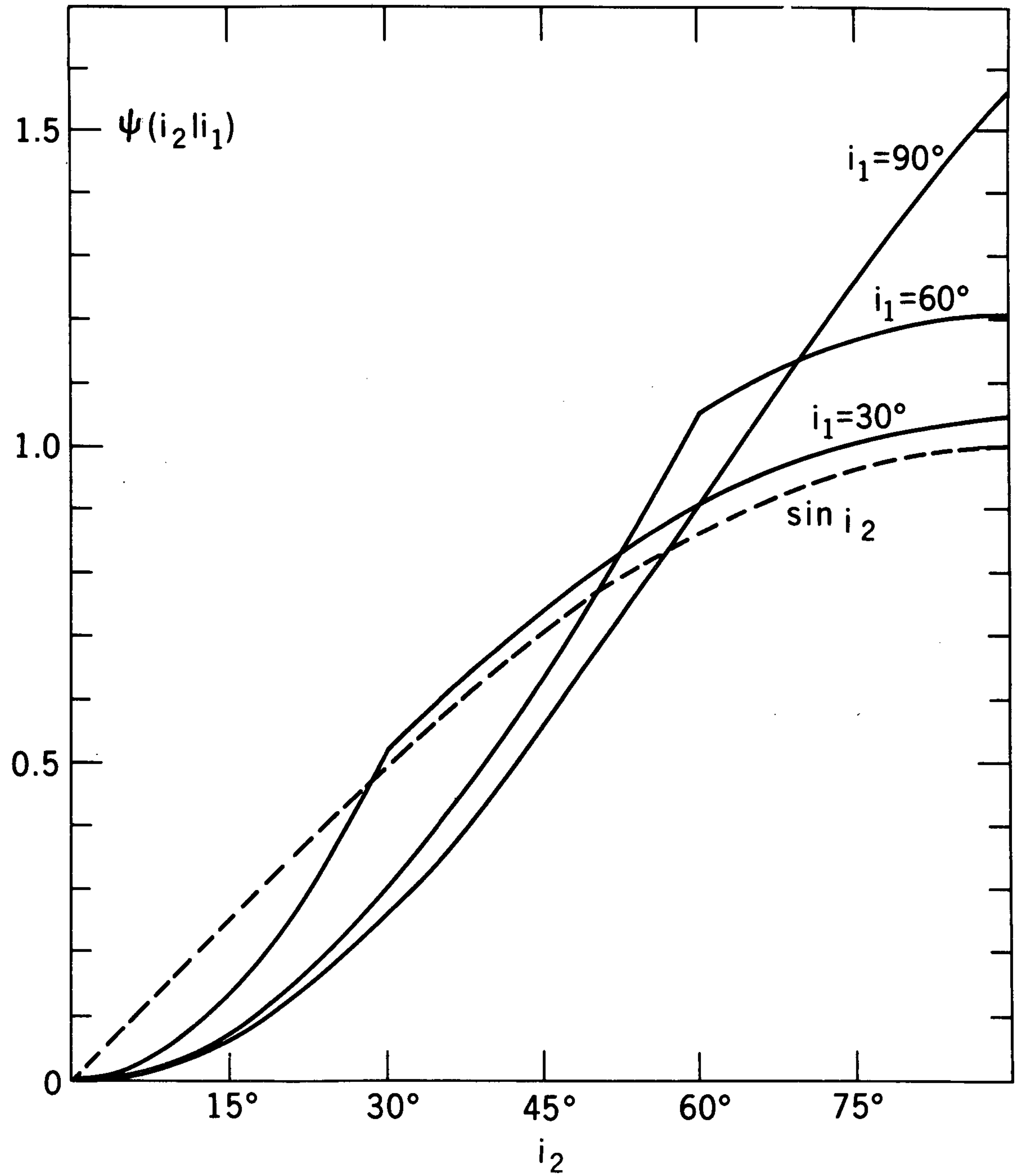
Fig. 4. The histogram of the apparent velocities $y_k = \underline{u}_k/\underline{v}_{ok}$ of the components of visual systems is shown with the statistical errors. It is assumed that no difference exists between the frequency distribution of the primaries and that of the secondaries. The dashed curve is the marginal distribution $H_k(x_k)$ of the true velocities $x_k = \underline{v}_k/\underline{v}_{ok}$. The solid curve is the computed distribution of the apparent velocities $F_k(y_k)$ which has been fitted to the histogram.

Fig. 5. Array distribution of $y_2$ for selected intervals of $y_1$ given in the form of conditioned histograms. The dashed curve represents the array frequency function $\Phi(x_2 \mid y_1)$ which produces the function $F(y_2 \mid y_1)$ (solid curves) considered to approximate the observations.

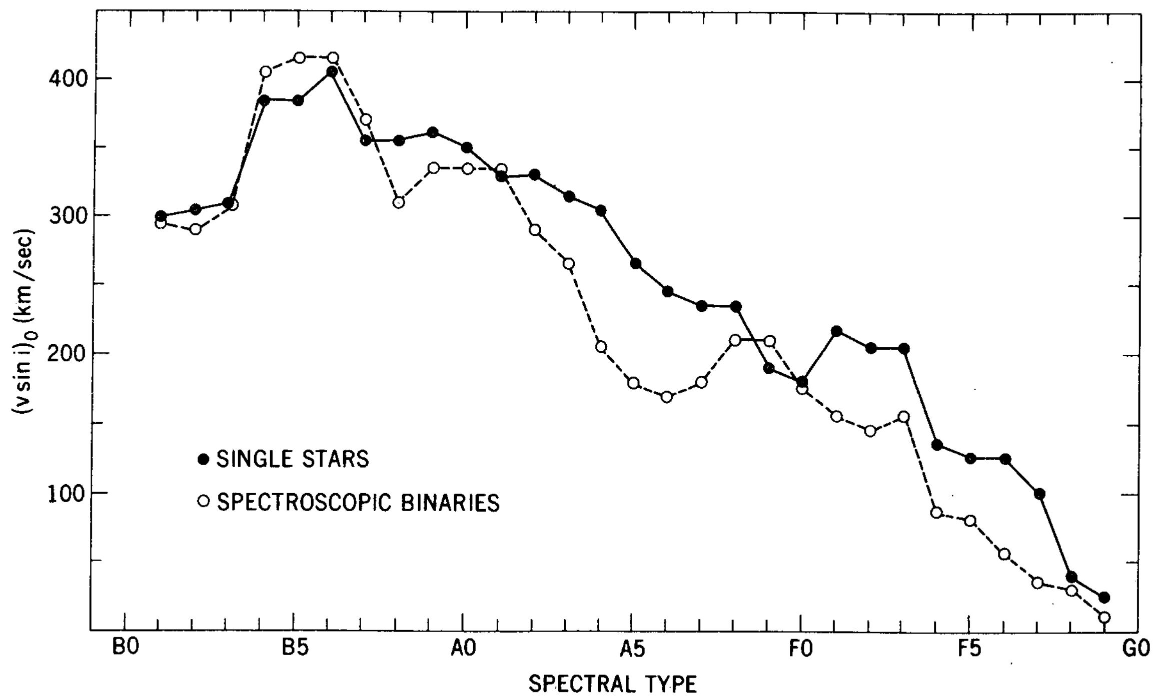
Fig. 6. The parameters of the array distribution $\Phi(x_2 \mid y_1)$ shown in Figure 5 are plotted <u>vs</u>. $y_1$ with the purpose to define the functional relationship $x_o(y_1)$ and $s(y_1)$. $x_o$ and $s$ are the mode and the dispersion of a normal law.

Fig. 7 The conditioned distribution $f(y_1 \mid x_2)$ (solid curves) computed by means of $f(y_1 \mid x_2)\ H_2(x_2) = F_1(y_1)\ \Phi(x_2 \mid y_1)$ is shown for the cases $x_2 = 0.1, 0.5, 0.9$. The frequency function $H(x_1 \mid x_2)$ is given by the dashed curve for $x_2 = 0.9$. The circles represent the function $f(y_1 \mid 0.9)$ computed through the integral equation relating $H(x_1 \mid x_2)$ and $f(y_1 \mid x_2)$. They approximate the same function $f(y_1 \mid x_2)$ independently derived.

Fig. 8. The correlation between $x_1$ and $x_2$ is described by the variation of the mode $x_{1M}$ and the dispersion $\sigma_1(x_2)$ of $H(x_1 \mid x_2)$ with $x_2$ along with the mean value of $x_1$, $\bar{x}_1$, as a function of $x_2$.

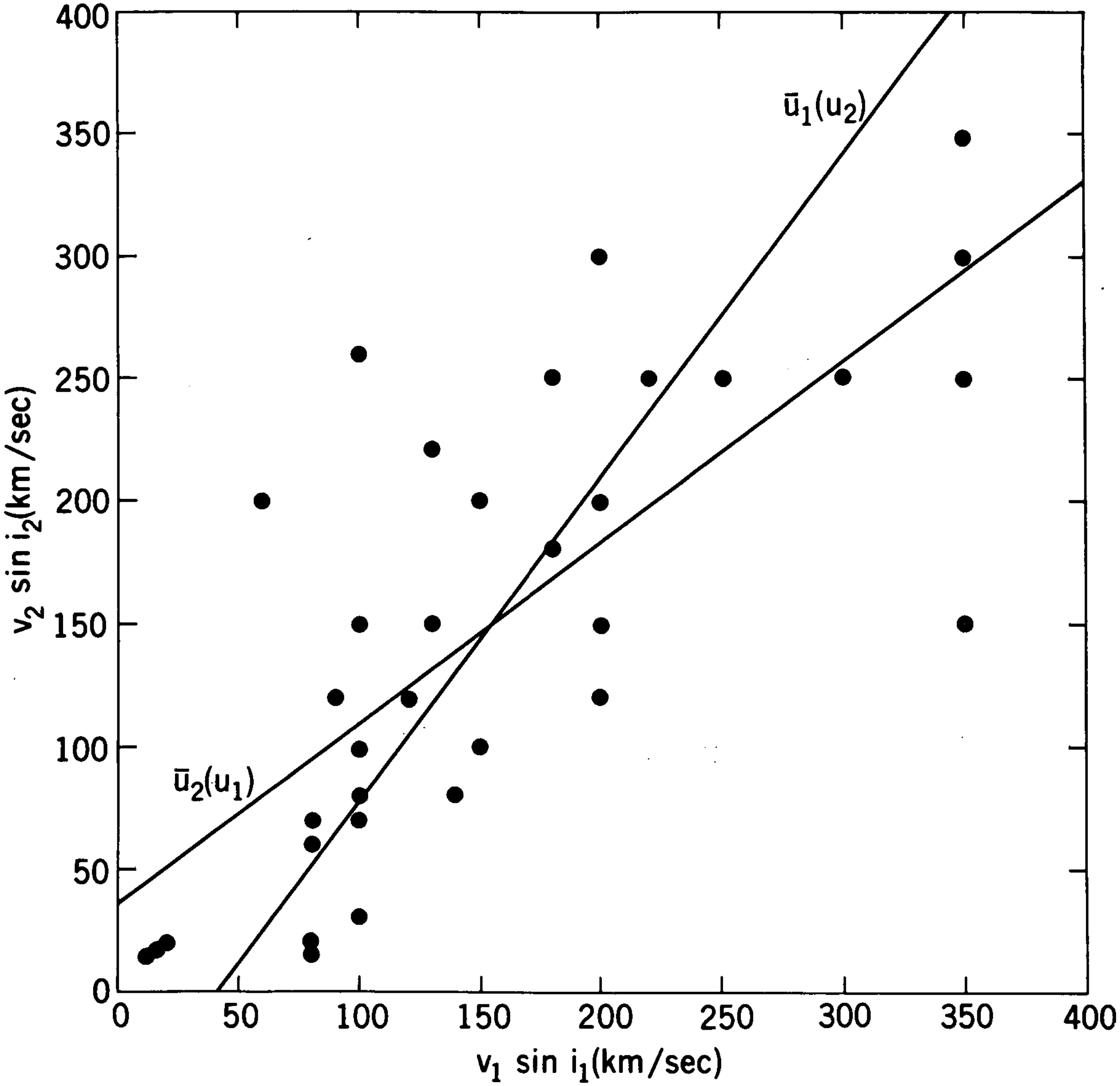


1.5
1.0
0.5
0
ψ(i₂|i₁)
i₁=90°
i₁=60°
i₁=30°
sin i₂
15°
30°
45°
60°
75°
i₂



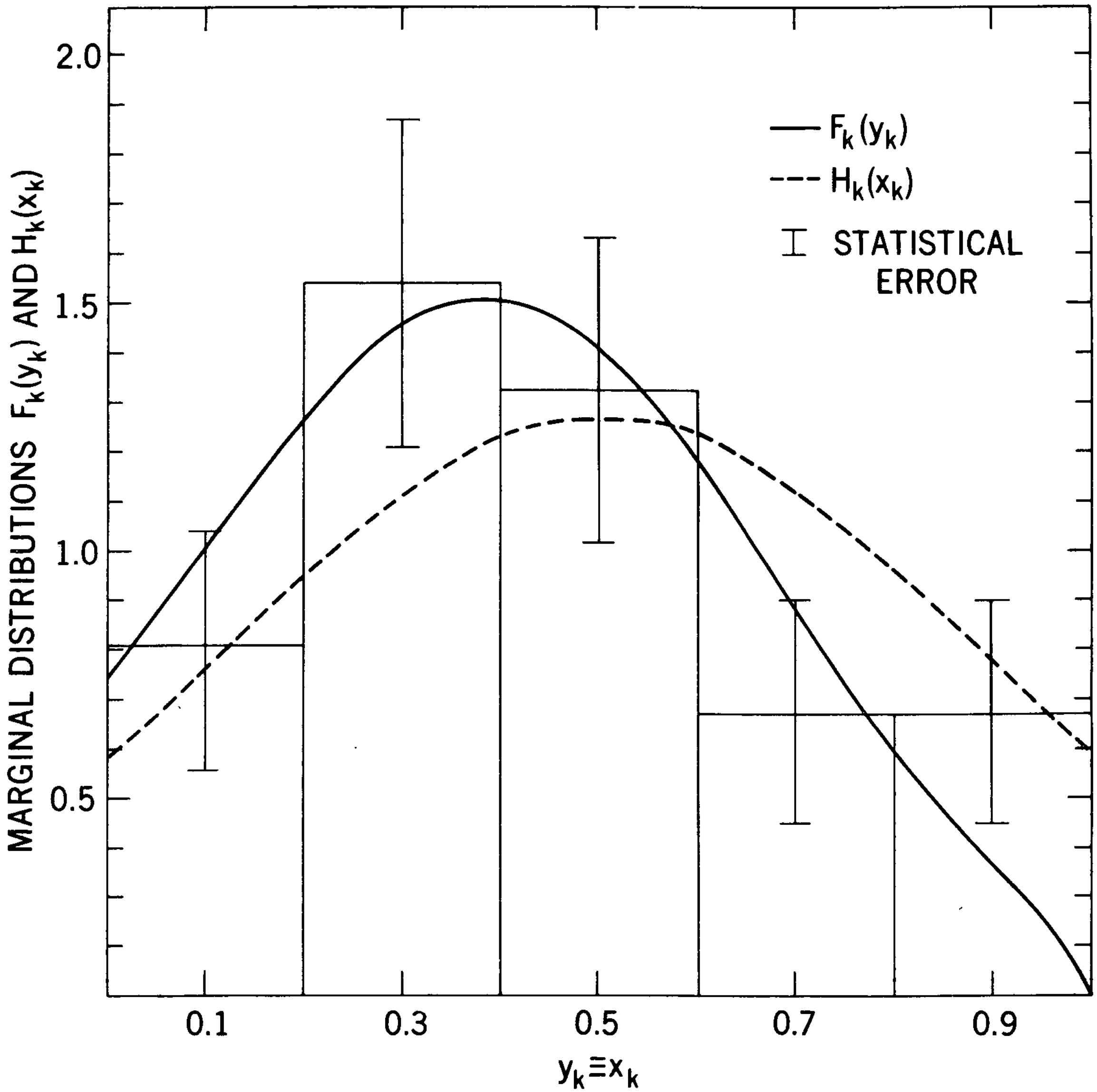
400
300
200
100
(v sin i)$_0$ (km/sec)
● SINGLE STARS
○ SPECTROSCOPIC BINARIES
B0
B5
A0
A5
F0
F5
G0
SPECTRAL TYPE


FIG. 2

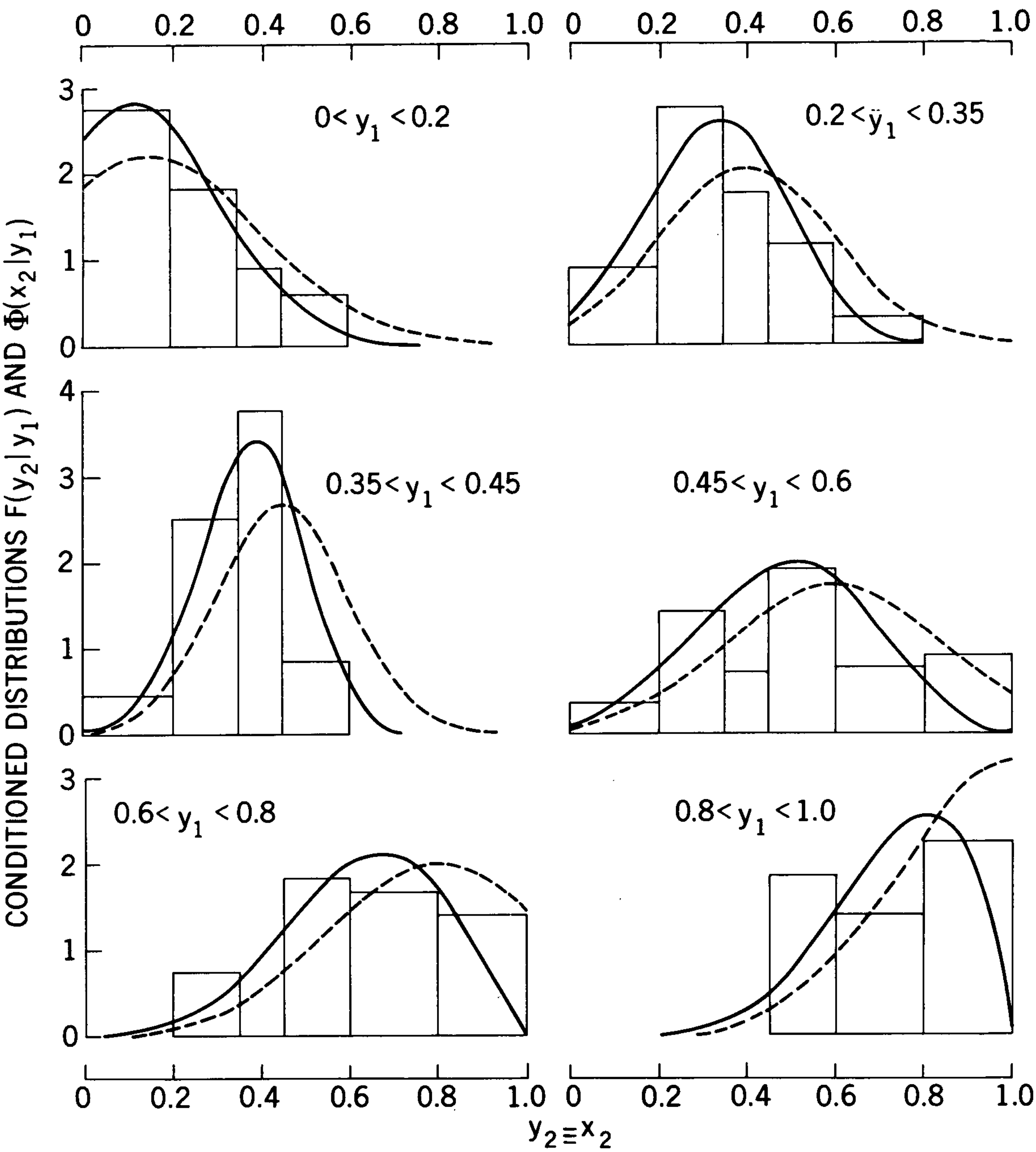


$\bar{u}_1(u_2)$
$\bar{u}_2(u_1)$
$v_2 \sin i_2$(km/sec)
$v_1 \sin i_1$(km/sec)
0
50
100
150
200
250
300
350
400

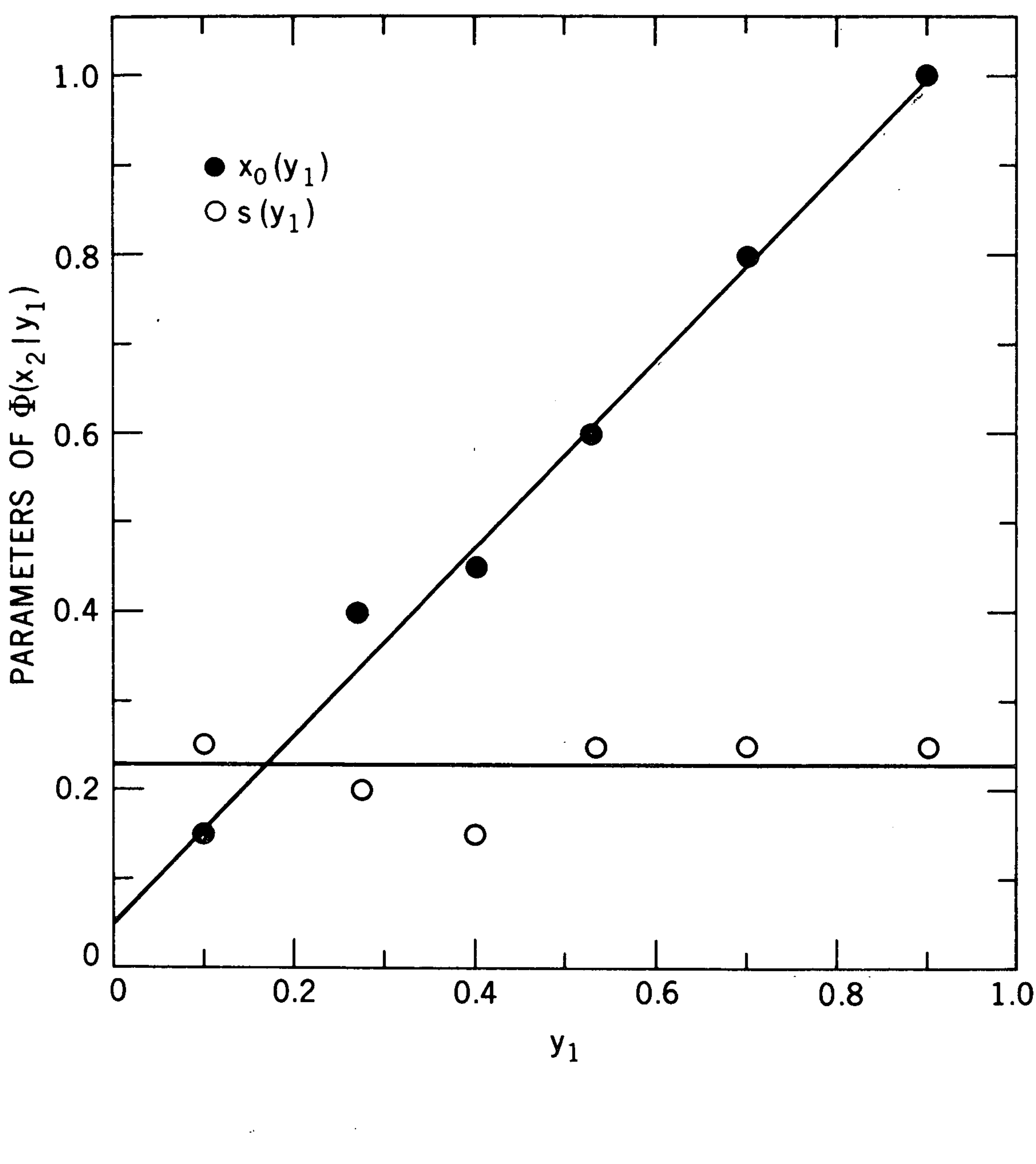

FIG. 3



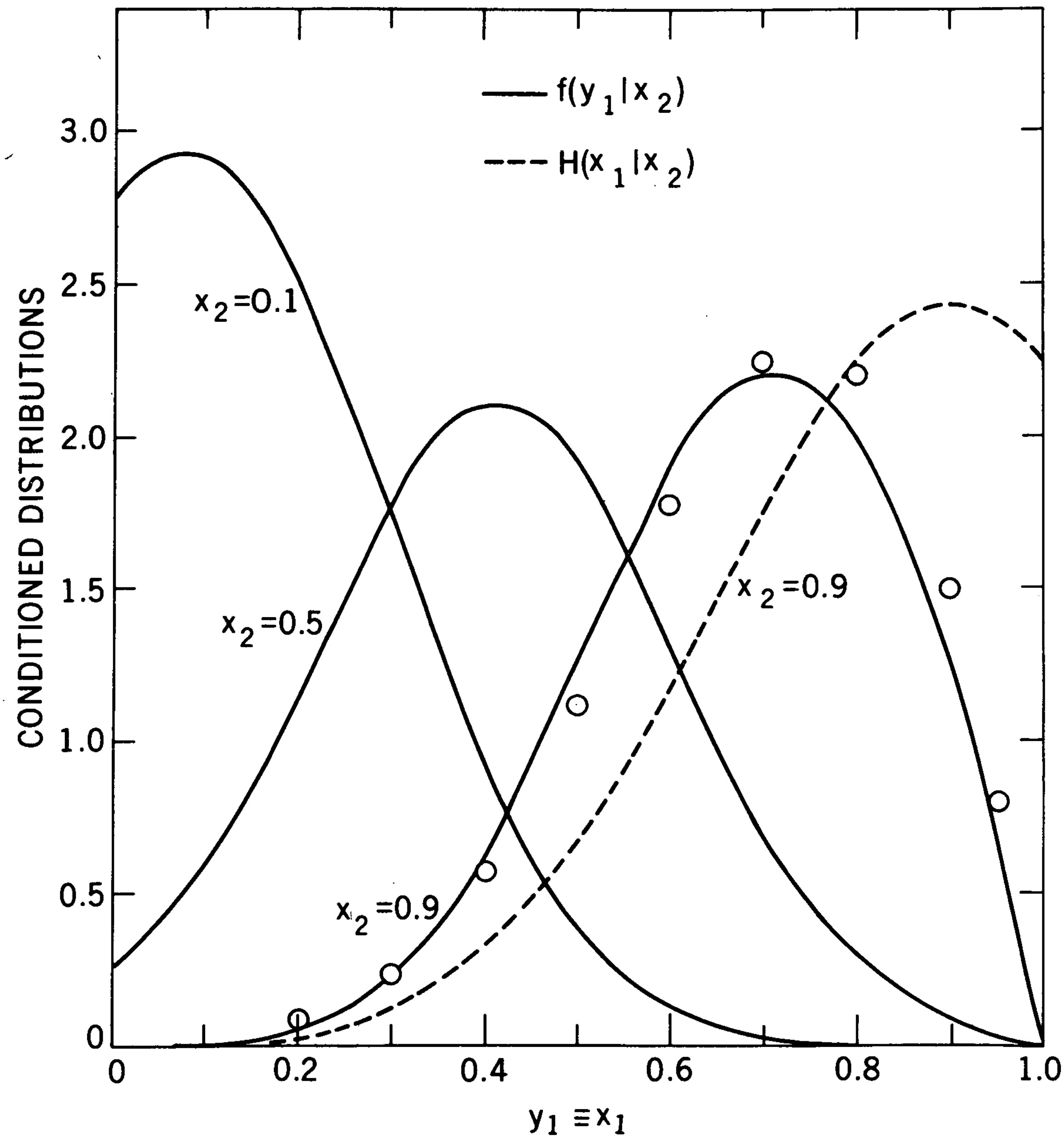
MARGINAL DISTRIBUTIONS $F_k(y_k)$ AND $H_k(x_k)$
2.0
1.5
1.0
0.5
0.1
0.3
0.5
0.7
0.9
$y_k \equiv x_k$
$F_k(y_k)$
$H_k(x_k)$
STATISTICAL ERROR



CONDITIONED DISTRIBUTIONS F(y2|y1) AND Φ(x2|y1)
0< y1 <0.2
0.2 <ÿ1 < 0.35
0.35< y1 < 0.45
0.45< y1 < 0.6
0.6< y1 <0.8
0.8< y1 <1.0
y2≡x2
0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4



$x_0(y_1)$
$s(y_1)$
PARAMETERS OF $\Phi(x_2 | y_1)$
$y_1$
0
0.2
0.4
0.6
0.8
1.0



f(y 1 | x 2)
H(x 1 | x 2)
x 2 =0.1
x 2 =0.5
x 2 =0.9
x 2 =0.9
CONDITIONED DISTRIBUTIONS
3.0
2.5
2.0
1.5
1.0
0.5
0
0
0.2
0.4
0.6
0.8
1.0
y1 ≡ x1

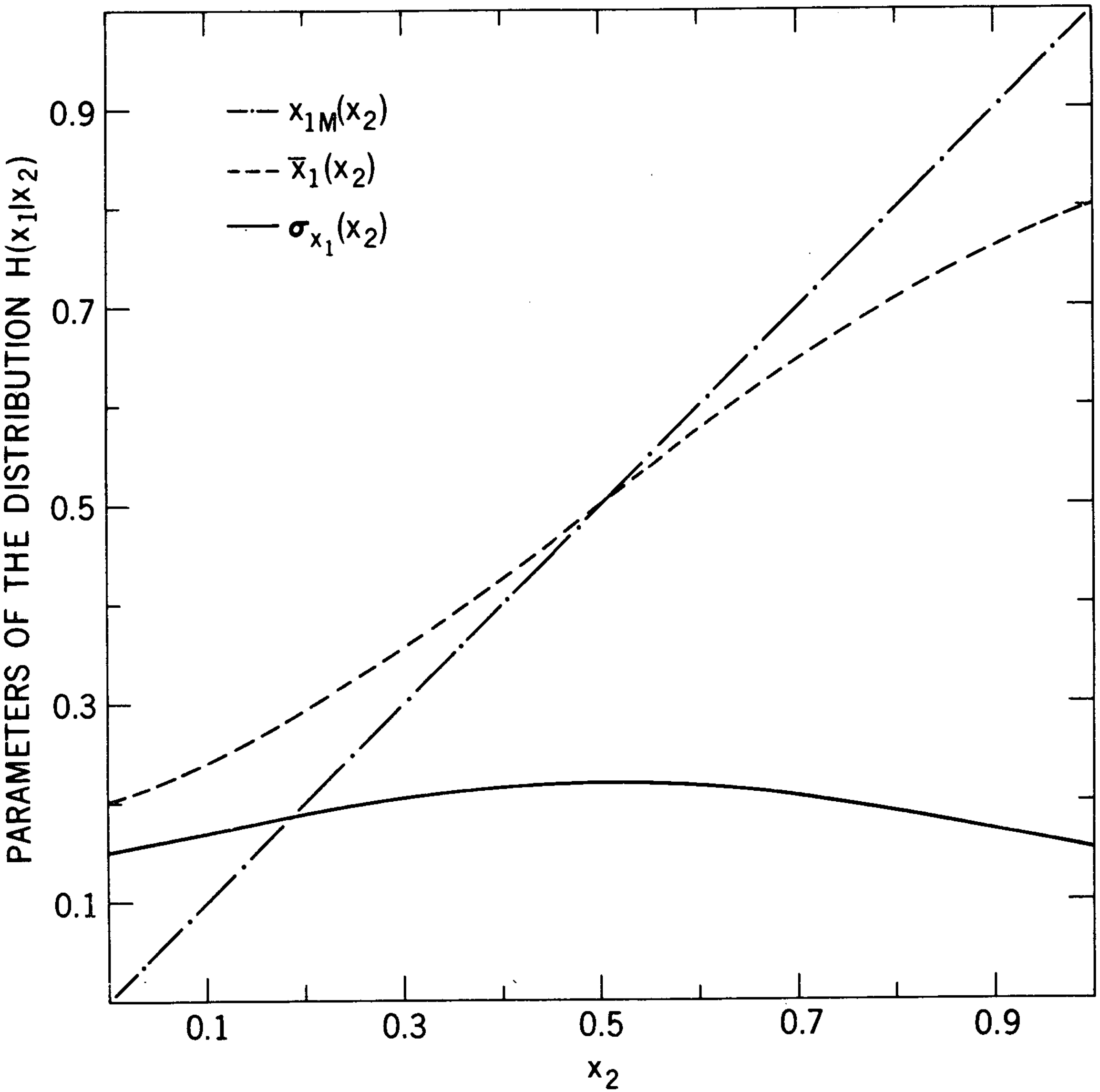

$x_{1M}(x_2)$
$\bar{x}_1(x_2)$
$\sigma_{x_1}(x_2)$
PARAMETERS OF THE DISTRIBUTION $H(x_1|x_2)$
0.9
0.7
0.5
0.3
0.1
0.1
0.3
0.5
0.7
0.9
$x_2$

FIG. 8